International Journal of Theoretical Physics, Vol. 40, No. 6, 2001

Autonomous Forms and Exact Solutions of Equations of Motion of Polytropic Gas

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Received July 27, 2000

The system of motion of a polytropic gas can be reduced to an autonomous form by using group analysis. A new family of exact solutions are constructed.

1. INTRODUCTION

In this paper, we look for invariant solutions that are particular exact solutions arising from symmetries of the equations of motion of a polytropic gas and characterized by means of the group analysis approach (see Olver, 1968; Ovsiannikov, 1982; Sedove, 1959). It has been shown (see Donato and Oliveri, 1993) that the nonautonomous first-order nonlinear partial differential equations admitting at least two one-parameter Lie groups of transformations with commuting infinitesimal operators (see Ames *et al.*, 1989; Donato, 1992) can be written in an autonomous form by a suitable use of the canonical variables. With reference to Donato and Oliveri (1993), it has been shown that the procedure can be applied to any kind of partial differential equations of any order if some suitable conditions are satisfied. Of course, one can also start from a system in the autonomous form admitting only trivial constant solutions, and by using appropriate canonical variables, one can transform it to a system that has nontrivial constant solutions that, in fact, are nonconstant in original variables. By using this procedure, we are able to build up new solutions of the equations of motion of a polytropic gas.

By considering Refs. (Doyle, 1999; Feinsilver *et al.*, 2000; Grigorycv *et al.*, 1999; Manganaro and Oliveri, 1989; Oliveri and Paola, 1999; Petrs, 1993), we see that in two dimensions, the equations governing the unsteady flow of a polylropic

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gas are given by

$$
\begin{aligned}
&\rho_t + u\rho_x + \nu\rho_y + \rho(u_x + \nu_y) = 0\\
&u_t + u_{u_x} + \nu u_y + (1/\rho)p_x = 0\\
&\nu_t + u_{\nu_x} + \nu\nu_y + (1/\rho)p_y = 0\\
&p_t + u_{p_x} + \nu p_y + \gamma p u_x = 0,
\end{aligned}
\tag{1.1}
$$

where ρ is the density, p the pressure, u and ν the velocity components in the x and y directions, respectively, and the adiabatic index γ is the ratio of the specific heats, generally a constant between 1 and $\frac{5}{3}$.

2. DETERMINATION OF LIE GROUPS

Classical Lie group theory is used to determine the classical symmetries of the system (1.1). The analysis was performed using the symmetry-finding software package DIMSYM (Sherring, 1993) making use of the symbolic manipulation package REDUCE (Hearn, 1991). The classical symmetries of the system (1.1) are given as follows:

$$
\begin{aligned}
X_1 &= \partial_t;\\
X_2 &= \partial_x;\\
X_3 &= \partial_y;\\
X_4 &= t\partial_x + \partial_u;\\
X_5 &= t\partial_y + \partial_\nu;\\
X_6 &= t\partial_t + x\partial_x + y\partial_y;\\
X_7 &= 2t\partial_t + x\partial_x + y\partial_y - u\partial_u - \nu\partial_\nu + 2\rho\partial_\rho;\\
X_8 &= y\partial_x - x\partial_y + \nu\partial_u - u\partial_\nu;\\
X_9 &= -y\partial_x + x\partial_y - \nu\partial_u + u\partial_\nu;\\
X_{10} &= \rho\partial_\rho + p\partial_p.
\end{aligned}
\tag{2.1}
$$

Under the operation of commutation, $[X_i, X_j] = X_i X_j - X_j X_i$, we can derive the following four cases:

Case 1. In this case, we can easily verify that the operators X_6 and X_{10} commute with X_7, that is,

$$[X_6, X_7] = 0 \quad \text{and} \quad [X_{10}, X_7] = 0. \tag{2.2}$$

Consequently, we write

$$[X_7, X_6 + kX_{10}] = 0, \tag{2.3}$$

where k is an arbitrary constant.

Upon following two infinitesimal operators related to the system (1.1)

$$\begin{aligned} E_1 &= X_7 = 2t\partial_t + x\partial_x + y\partial_y - u\partial_u - v\partial_v + 2\rho\partial_\rho, \\ E_2 &= X_6 + kX_{10} = t\partial_t + x\partial_x + y\partial_y + kp\partial_p + k\rho\partial_\rho. \end{aligned} \tag{2.4}$$

In order to write the system (1.1) in the autonomous form, we choose a suitable condition by introducing the following canonical variables:

$$\begin{aligned} &E_1 T = 1, \quad E_1 U = 0, \quad E_1 P_0 = 0, \\ &E_1 \xi = 0, \quad E_1 R = 0, \\ &E_1 \eta = 0, \quad E_1 H = 0. \end{aligned} \tag{2.5}$$

Thus, the infinitesimal operator E_1 is converted to a translation in T with these canonical variables. That is,

$$\tilde{E}_1 = \frac{\partial}{\partial T}. \tag{2.6}$$

By integrating the system (2.5), one can see that a possible choice of the canonical variables leads to

$$\begin{aligned} &T = \frac{1}{2}\ln t, \qquad v = Rx^{-1}, \\ &\xi = xt^{-1/2}, \quad \rho = Hx^2, \\ &\eta = yt^{-1/2}, \quad p = P_0. \\ &u = Ux^{-1}, \end{aligned} \tag{2.7}$$

In terms of the transformation of variables, we are able to write the system (1.1) in the form

$$\frac{\partial V}{\partial T} + \tilde{A}(V)\frac{\partial V}{\partial \xi} + \tilde{B}(V)\frac{\partial V}{\partial \eta} = 0, \tag{2.8}$$

where

$$V = \begin{bmatrix} H \\ U \\ R \\ P_0 \end{bmatrix}, \quad \tilde{A}(V) = \begin{bmatrix} U & H & 0 & 0 \\ 0 & U & 0 & H^{-1} \\ 0 & 0 & U & 0 \\ 0 & P_0 & 0 & U \end{bmatrix}, \quad \tilde{B}(V) = \begin{bmatrix} R & 0 & H & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R & H^{-1} \\ 0 & 0 & \gamma P_0 & R \end{bmatrix}.$$

The infinitesimal operator E_2 can be written in terms of the canonical variables and takes the following form:

$$E_2 = T\frac{\partial}{\partial T} + \xi\frac{\partial}{\partial \xi} + \eta\frac{\partial}{\partial \eta} + kH\frac{\partial}{\partial H} + kP_0\frac{\partial}{\partial P_0}. \tag{2.9}$$

Now, we can investigate new canonical variables $\tilde{T}$, $\tilde{\xi}$, $\tilde{H}$, $\tilde{P}_0$, $\tilde{U}$, $\tilde{R}$, and $\tilde{\eta}$ related to the operator $\tilde{E}_2$ and defined by

$$\begin{aligned}
&\tilde{E}_2\tilde{T} = 1, \quad \tilde{E}_2\tilde{U} = 0, \quad \tilde{E}_2\tilde{P}_0 = 0,\\
&\tilde{E}_2\tilde{\xi} = 0, \quad \tilde{E}_2\tilde{R} = 0,\\
&\tilde{E}_2\tilde{\eta} = 0, \quad \tilde{E}_2\tilde{H} = 0.
\end{aligned} \tag{2.10}$$

Consequently, we get the new transformation of variables in the form

$$\begin{aligned}
\tilde{T} &= \ln T, \qquad P_0 = \tilde{P}_0 T^k,\\
\tilde{\xi} &= \xi T^{-1}, \quad U = \tilde{U}(T, \xi, \eta),\\
\tilde{\eta} &= \eta T^{-1}, \quad R = \tilde{R}(T, \xi, \eta),\\
H &= \tilde{H} T^k.
\end{aligned} \tag{2.11}$$

Finally, we obtain the following system in the autonomous form

$$\frac{\partial W}{\partial T} + \tilde{\tilde{A}}(W)\frac{\partial W}{\partial \xi} + \tilde{\tilde{B}}(W)\frac{\partial W}{\partial \tilde{\eta}} = 0, \tag{2.12}$$

where

$$W = \begin{bmatrix} \tilde{H} \\ \tilde{U} \\ \tilde{R} \\ \tilde{P}_0 \end{bmatrix}, \qquad \tilde{\tilde{A}}(W) = \begin{bmatrix} \tilde{U} & \tilde{H} & 0 & 0 \\ 0 & \tilde{U} & 0 & \tilde{H}^{-1} \\ 0 & 0 & \tilde{U} & 0 \\ 0 & \gamma\tilde{P}_0 & 0 & \tilde{U} \end{bmatrix},$$

$$\tilde{\tilde{B}}(W) = \begin{bmatrix} \tilde{R} & 0 & \tilde{H} & 0 \\ 0 & \tilde{R} & 0 & 0 \\ 0 & 0 & \tilde{R} & \tilde{H}^{-1} \\ 0 & 0 & \gamma\tilde{P}_0 & \tilde{R} \end{bmatrix}.$$

3. SOME CLASSES OF PARTICULAR SOLUTIONS

Utilizing the preceding results we can find that the system (1.1) is converted to the system (2.12) by the following transformation, which is obtained by joining

(2.7) and (2.11):

$$\begin{aligned}
\tilde{T} &= \ln\left[\frac{1}{2}\ln T\right], & p &= \tilde{p}_0\left[\frac{1}{2}\ln T\right]^k, \\
\tilde{\xi} &= (xt^{-1/2})\left[\frac{1}{2}\ln T\right]^{-1}, & u &= \tilde{U}x^{-1}, \\
\tilde{\eta} &= (yt^{-1/2})\left[\frac{1}{2}\ln T\right]^{-1}, & v &= \tilde{R}x^{-1}, \\
\rho &= \tilde{H}x^2\left[\frac{1}{2}\ln T\right]^k. &&
\end{aligned} \tag{3.1}$$

In order to build up a particular solution, by inspection, we select a suitable assumption to distinguish various cases.

(I) $k = 0,\ W = W(\tilde{\eta}),\ \tilde{U} = 0$

Returning to the system (2.12), and taking into consideration (3.1), we obtain the system

$$\begin{aligned}
[\tilde{R}\tilde{H}]' &= 0, \\
\tilde{R}\tilde{R}' + \tilde{H}^{-1}p_0' &= 0, \\
\gamma\tilde{p}_0 R' + \tilde{R}p_0' &= 0,
\end{aligned} \tag{3.2}$$

where the prime ($'$) denotes $\partial/\partial\tilde{\eta}$. This system leads to the solution

$$\begin{aligned}
\tilde{U} &= 0, \\
\tilde{R} &= \frac{\gamma Z}{(\gamma+1)B}, \\
\tilde{p}_0 &= \frac{\gamma Z}{(\gamma+1)}, \\
\tilde{H} &= \frac{(1+\gamma)B^2}{\gamma Z},
\end{aligned}$$

where B and Z are nonzero constants.

Going back to the system (3.1), we see that the corresponding solutions are given by

$$\begin{aligned}
u &= 0, \quad v = \frac{\gamma Z}{(\gamma+1)B}x^{-1}, \\
p &= \frac{Z}{(\gamma+1)}, \quad \rho = \frac{(\gamma+1)B^2}{\gamma Z}x^2.
\end{aligned}$$

(II) $W = \tilde{W}(\tilde{T})$ or $W = \tilde{W}(\tilde{\xi})$

In this case we can easily verify that the solutions for $\tilde{U}$, $\tilde{R}$, $\tilde{P}$, and $\tilde{H}$ have the forms

$$\begin{aligned} \tilde{U} &= 0, & \tilde{P} &= Z. \\ \tilde{R} &= Q, & \tilde{H} &= B. \end{aligned}$$

The corresponding solutions are

$$\begin{aligned} u &= 0, & \nu &= Qx^{-1}, \\ \rho &= Bx^2, & p &= Z, \end{aligned}$$

where B, Q, and Z are constants.

Case 2. We investigate another class of solutions by using another representation of canonical variables which leads to

$$\begin{aligned} T &= \frac{1}{2}\ln t, & \nu &= Ry^{-1}, \\ \xi &= xt^{-1/2}, & \rho &= Hy^2, \\ \eta &= yt^{-1/2}, & p &= P_0(T, \xi, \eta), \\ u &= Uy^{-1}. \end{aligned} \tag{3.3}$$

This represents a transformation of variables allowing to write the system (1.1) in the same form (2.8). Moreover, let us introduce the infinitesimal operator E_2 related to the canonical variables (3.3) in the form

$$E_2 = T\frac{\partial}{\partial T} + \xi\frac{\partial}{\partial \xi} + \eta\frac{\partial}{\partial \eta} + kH\frac{\partial}{\partial H} + kP_0\frac{\partial}{\partial P_0}. \tag{3.4}$$

Then the new canonical variables joining to the operator E_2 can be expressed as follows:

$$\begin{aligned} &\tilde{E}_2\tilde{T} = 1, \quad \tilde{E}_2\tilde{U} = 0, \quad \tilde{E}_2\tilde{P}_0 = 0, \\ &\tilde{E}_2\tilde{\xi} = 0, \quad \tilde{E}_2\tilde{R} = 0, \\ &\tilde{E}_2\tilde{\eta} = 0, \quad \tilde{E}_2\tilde{H} = 0, \end{aligned} \tag{3.5}$$

whereupon, it is possible to obtain the new transformation of variables

$$\begin{aligned} \tilde{T} &= \ln T, & P_0 &= \tilde{P}_0T^k, \\ \tilde{\xi} &= \xi T^{-1}, & U &= \tilde{U}(\tilde{T}, \tilde{\xi}, \tilde{\eta}), \\ \tilde{\eta} &= \eta T^{-1}, & R &= \tilde{R}(\tilde{T}, \tilde{\xi}, \tilde{\eta}), \\ H &= \tilde{H}T^k. \end{aligned} \tag{3.6}$$

Consequently, the autonomous form is obtained and has the same expression (2.12).

4. CLASS OF PARTICULAR SOLUTIONS

In order to find the solution in this case the general canonical variables have the following form

$$\begin{aligned}
\tilde{T} &= \ln\left[\frac{1}{2}\ln t\right], & p &= \tilde{P}_0\left[\frac{1}{2}\ln t\right]^k, \\
\tilde{\xi} &= \left(xt^{-1/2}\right)\left[\frac{1}{2}\ln t\right]^{-1}, & u &= \tilde{U}y^{-1}, \\
\tilde{\eta} &= \left(yt^{-1/2}\right)\left[\frac{1}{2}\ln T\right]^{-1}, & v &= \tilde{R}y^{-1}, \\
\rho &= \tilde{H}y^2\left[\frac{1}{2}\ln t\right]^k. & &
\end{aligned} \tag{4.1}$$

It follows from the inspection choice that

$$k = 0, \qquad W = W(\tilde{\xi}), \qquad \tilde{R} = 0.$$

Hence system (2.12) becomes

$$\begin{aligned}
[\tilde{U}\tilde{H}]' &= 0, \\
\tilde{U}\tilde{U}' + \tilde{H}^{-1}p_0' &= 0, \\
\gamma\tilde{p}_0\tilde{U}' + \tilde{U}p_0' &= 0,
\end{aligned} \tag{4.2}$$

where the prime ($'$) denotes $\partial/\partial\tilde{\xi}$.

Exact solution for the system (4.2) can be obtained by some calculations as follows:

$$\begin{aligned}
\tilde{R} &= 0, \\
\tilde{U} &= \frac{\gamma Z}{(\gamma+1)B}, \\
\tilde{p}_0 &= \frac{Z}{(\gamma+1)}, \\
\tilde{H} &= \frac{(1+\gamma)B^2}{\gamma Z}.
\end{aligned}$$

In terms of the original variables, we obtain

$$\begin{aligned}
v &= 0, & u &= \frac{\gamma Z}{(\gamma+1)B}y^{-1}, \\
p &= \frac{Z}{(\gamma+1)}, & \rho &= \frac{(\gamma+1)B^2}{\gamma Z}y^2.
\end{aligned}$$

Case 3. In this case, we observe that both X_7 and X_4 commute with X_6, that is,

$$[X_6, X_7] = 0 \quad \text{and} \quad [X_6, X_4] = 0.$$

Consequently, we write

$$[X_6, X_7 + k\ X_4] = 0.$$

Thus, we can see that E_1 and E_2 are two infinitesimal operators admitted by the system (1.1) as the following:

$$\begin{aligned} \mathrm{E}_1 &= X_7 + kX_4 = 2t\partial_t + (kt + x)\partial_x + y\partial_y + (k-u)\partial_u - \nu\partial_\nu + 2\rho\partial_\rho, \\ \mathrm{E}_2 &= X_6 = t\partial_t + x\partial_x + y\partial_y. \end{aligned}$$

The canonical variables T, ξ, η, U, R, H, and P_0 in terms of the operator E_1 are given by

$$\begin{aligned} E_1 T &= 1, \quad E_1 U = 0, \quad E_1 P_0 = 0, \\ E_1 \xi &= 0, \quad E_1 R = 0, \\ E_1 \eta &= 0, \quad E_1 H = 0. \end{aligned} \tag{4.3}$$

By integrating the system (4.3) one can see that a possible choice of the canonical variables leads to

$$\begin{aligned} T &= \frac{1}{2}\ln t, & u &= k - Uy^{-1}, \\ \xi &= [ke^{2t} + x]t^{-1/2}, & \nu &= Ry^{-1}, \\ \eta &= yt^{-1}, & \rho &= Hy^2, \\ p &= P_0. \end{aligned} \tag{4.4}$$

We note that this representation allows us to write the system (1.1) in the same form (2.8). Moreover, in terms of the canonical variables (4.3), the infinitesimal operator E_2 assumes the form

$$E_2 = T\partial_T + \xi\partial_\xi + \eta\partial_\eta.$$

Now, the new canonical variables are in the form

$$\begin{aligned} \tilde{T} &= \ln T, & P_0 &= \tilde{P}_0(\tilde{T}, \tilde{\xi}, \tilde{\eta}), \\ \tilde{\xi} &= \xi T^{-1}, & U &= \tilde{U}(\tilde{T}, \tilde{\xi}, \tilde{\eta}), \\ \tilde{\eta} &= \eta T^{-1}, & R &= \tilde{R}(\tilde{T}, \tilde{\xi}, \tilde{\eta}), \\ H &= \tilde{H}(\tilde{T}, \tilde{\xi}, \tilde{\eta}), \end{aligned} \tag{4.5}$$

and

$$\tilde{E}_2\tilde{T} = 1, \quad \tilde{E}_2 U = 0, \quad \tilde{E}_2 P_0 = 0,$$
$$\tilde{E}_2\tilde{\xi} = 0, \quad \tilde{E}_2 R = 0,$$
$$\tilde{E}_2\tilde{\eta} = 0, \quad \tilde{E}_2 H = 0.$$

Finally, we obtain the same form (2.12).

5. SOME CLASSES OF PARTICULAR SOLUTIONS

We work on the transformation which are obtained from the systems (4.4) and (4.5) of the form

$$\begin{aligned}
\tilde{T} &= \ln\left[\frac{1}{2}\ln t\right], & p &= \tilde{p}_0,\\
\tilde{\xi} &= [ke^{2t} + x]t^{-1/2}\left[\frac{1}{2}\ln t\right]^{-1}, & u &= k - \tilde{U}y^{-1},\\
\tilde{\eta} &= \left(yt^{-1/2}\right)\left[\frac{1}{2}\ln T\right]^{-1}, & v &= \tilde{R}y^{-1},\\
\rho &= \tilde{H}y^2.
\end{aligned} \tag{5.1}$$

We discuss some particular solutions as

(i) If $k \neq 0$, $W = W(\tilde{\xi})$, $\tilde{R} = 0$.

After some calculations, we get

$$\tilde{R} = 0,$$
$$\tilde{U} = \frac{\gamma Z}{(\gamma+1)B},$$
$$\tilde{P}_0 = \frac{Z}{(\gamma+1)},$$
$$\tilde{H} = \frac{(1+\gamma)B^2}{\gamma Z}.$$

The corresponding solutions of the original system are

$$u = k - \frac{\gamma Z}{(\gamma+1)B}y^2, \quad v = 0,$$
$$p = \frac{Z}{(\gamma+1)}, \qquad \rho = \frac{(\gamma+1)B^2}{\gamma Z}y^2.$$

(ii) If $W = W(\tilde{T})$, $\tilde{R} = 0$.

In this case we find that

$$\tilde{H} = A, \quad \tilde{U} = B,$$
$$\tilde{P} = E, \quad \tilde{R} = 0,$$

where A, B, and E are arbitrary constants. Then the solutions of the main system are

$$\begin{aligned} u &= k - By^{-1}, \quad & \rho &= Ay^2, \\ v &= 0, & p &= E. \end{aligned}$$

Case 4. In this case, we can infer that both X_8 and X_9 commute with X_7. Consequently, $[X_7, X_8 + kX_9] = 0$. We may then write E_1 and E_2 as follows:

$$\begin{aligned} E_1 &= 2t\partial_t + x\partial_x + y\partial_y - u\partial_u - v\partial_v + 2\rho\partial_\rho; \\ E_2 &= (1-k)y\partial_x - (1-k)x\partial_y + (1-k)v\partial_u - (1-k)u\partial_v. \end{aligned} \tag{5.2}$$

The analysis of the system (5.2) and the canonical variables (4.3) corresponding to operator E_1 has disclosed a reduction to the system as described by

$$\begin{aligned} T &= \frac{1}{2}\ln t, \quad & u &= Ut^{-1/2}, \\ \xi &= xt^{-1/2}, & v &= Rt^{-1/2}, \\ \eta &= yt^{-1/2}, & \rho &= Hx^2, \\ p &= p_0. \end{aligned} \tag{5.3}$$

From the last result we can rearrange the system (1.1) in the form (2.8). Now, we work on the operator E_2 as before. We introduce the canonical variables related to operator E_2 and defined by

$$\begin{aligned} \tilde{E}_2\tilde{T} &= 1, \quad \tilde{E}_2 U = 0, \quad \tilde{E}_2 P_0 = 0, \\ \tilde{E}_2\tilde{\xi} &= 0, \quad \tilde{E}_2 R = 0, \\ \tilde{E}_2\tilde{\eta} &= 0, \quad \tilde{E}_2 H = 0. \end{aligned} \tag{5.4}$$

Whereupon we can calculate the new transformation variables

$$\begin{aligned} \tilde{\eta} &= \xi^2 + \eta^2, \\ \tilde{\xi} &= \frac{1}{(1-k)\sqrt{\tilde{\eta}}}\ \sin^{-1}\left(\frac{\xi}{\sqrt{\tilde{\eta}}}\right), \\ \phi &= U^2 + R^2, \end{aligned}$$

$$
\begin{aligned}
U &= \sqrt{\phi}\ \sin\sqrt{\phi}\left[\frac{1}{\sqrt{\tilde{\eta}}}\ \sin^{-1}\left(\frac{\xi}{\sqrt{\tilde{\eta}}}\right) + \tilde{U}\right],\\
R &= \sqrt{\phi}\ \cos\sqrt{\phi}\left[\frac{1}{\sqrt{\tilde{\eta}}}\ \sin^{-1}\left(\frac{\xi}{\sqrt{\tilde{\eta}}}\right) + \tilde{R}\right],\\
T &= \tilde{T},\\
H &= \tilde{H},\\
P &= \tilde{P}_0.
\end{aligned}
$$

Then we obtain the same form (2.12). The transformation linking the original system (1.1) and the system (2.12) is

$$
\begin{aligned}
\tilde{T} &= \frac{1}{2}\ \ln t,\\
\tilde{\xi} &= \frac{1}{(1-k)\sqrt{[x^2+y^2]}}\ \sin^{-1}\left(\frac{xt^{-1}}{\sqrt{[x^2+y^2]\,t^{-1}}}\right),\\
U &= \sqrt{\phi}\ \sin\sqrt{\phi}\left[\frac{1}{\sqrt{[x^2+y^2]\,t^{-1}}}\ \sin^{-1}\left(\frac{xt^{-1}}{\sqrt{[x^2+y^2]\,t^{-1}}} + \tilde{U}\right)\right]t^{-1},\\
R &= \sqrt{\phi}\ \cos\sqrt{\phi}\left[\frac{1}{\sqrt{[x^2+y^2]\,t^{-1}}}\ \sin^{-1}\left(\frac{xt^{-1}}{\sqrt{[x^2+y^2]\,t^{-1}}}\right) + \tilde{R}\right]t^{-1},\\
p &= \tilde{P}_0,\\
\rho &= \tilde{H}x^2,\\
\tilde{\eta} &= [x^2+y^2]t^{-1}.
\end{aligned}
$$

6. CLASS OF SOLUTIONS

If $k \neq 0$, $W = W(\xi)$. Then we get

$$
\begin{aligned}
[\tilde{U}\tilde{H}]' &= 0,\\
\tilde{U}\tilde{R}' &= 0,\\
\tilde{U}\tilde{U}' + \tilde{H}^{-1}p_0' &= 0,\\
\gamma\,\tilde{p}_0\tilde{U}' + \tilde{U}p_0' &= 0,
\end{aligned}
$$

where the prime (′) denotes $\partial/\partial\tilde{\xi}$.

The solution of the last system is

$$\tilde{U} = E, \quad \tilde{H} = F,$$
$$\tilde{P}_0 = Z, \quad \tilde{R} = A.$$

We obtain the solution of the original system (1.1) in the form

$$u = \sqrt{\phi}\ \sin\sqrt{\phi}\left[\frac{1}{\sqrt{[x^2+y^2]\,t^{-1}}}\ \sin^{-1}\left(\frac{xt^{-1}}{\sqrt{[x^2+y^2]\,t^{-1}}}\right) + E\right]t^{-1},$$
$$\nu = \sqrt{\phi}\ \cos\sqrt{\phi}\left[\frac{1}{\sqrt{[x^2+y^2]\,t^{-1}}}\ \sin^{-1}\left(\frac{xt^{-1}}{\sqrt{[x^2+y^2]\,t^{-1}}}\right) + A\right]t^{-1},$$
$$p = Z,$$
$$\rho = Fx^2,$$

where ϕ, Z, E, A, and F are arbitrary constants.

Case 5. In this case we find that X_2 and X_5 commute with X_3. Consequently, $[X_3, X_2 + kX_5] = 0$. Thus we obtain the following two infinitesimal operators admitted by the system (1.1)

$$E_1 = \partial y,$$
$$E_2 = \partial_x + tk\ \partial_y + k\ \partial_\nu.$$

According to the algorithm in the last case, we find that the canonical variables related to the infinitesimal operator E_1 are defined by

$$\begin{aligned} E_1T &= 0, \quad E_1U = 0, \quad E_1P_0 = 0,\\ E_1\xi &= 0, \quad E_1R = 0,\\ E_1\eta &= 0, \quad E_1H = 0. \end{aligned} \tag{6.1}$$

and the infinitesimal operator E_1 can be written in the form

$$\tilde{E}_1 = \partial_\eta.$$

It can be shown by integration that the system (6.1) is reduced to the following system:

$$\begin{aligned} T &= t, & p &= P_0(T, \xi, \eta),\\ \xi &= x, & u &= U(T, \xi, \eta),\\ \eta &= y, & \nu &= R(T, \xi, \eta),\\ \rho &= H(T, \xi, \eta). \end{aligned} \tag{6.2}$$

Consequently, the system (1.1) can be written in the form (2.8). Taking into consideration that the operator E_2 and the canonical variables can be represented in the form

$$\begin{aligned}
&\tilde{E}_2\tilde{T} = 0, \quad \tilde{E}_2 U = 0, \quad \tilde{E}_2 P_0 = 0,\\
&\tilde{E}_2\tilde{\xi} = 1, \quad \tilde{E}_2 R = 0,\\
&\tilde{E}_2\tilde{\eta} = 0, \quad \tilde{E}_2 H = 0,
\end{aligned} \tag{6.3}$$

we can write the new transformation of variables as follows:

$$\begin{aligned}
&\tilde{T} = T, && P_0 = \tilde{P}_0(\tilde{T}, \tilde{\xi}, \tilde{\eta}),\\
&\tilde{\xi} = \xi, && U = \tilde{U}(\tilde{T}, \tilde{\xi}, \tilde{\eta}),\\
&\tilde{\eta} = \eta - \tilde{T}k\xi, && H = \tilde{H}(\tilde{T}, \tilde{\xi}, \tilde{\eta}),\\
&\tilde{R} = R + k\xi,
\end{aligned} \tag{6.4}$$

and hence, we get the system in the autonomous form as in (2.12). The relation between (6.2) and (6.4) leads to

$$\begin{aligned}
&\tilde{T} = t, && p = \tilde{P}_0,\\
&\tilde{\xi} = x, && u = \tilde{U},\\
&\tilde{\eta} = y - ktx, && v = \tilde{R} + kx,\\
&\rho = \tilde{H}.
\end{aligned} \tag{6.5}$$

7. CLASS OF SOME SOLUTIONS

We can obtain special solutions such as

(i) $k = 0$, $W = W(\tilde{T})$,

for which we have the solutions

$$u = c_1, \qquad v = c_2, \qquad p = c_3, \qquad \rho = c_4.$$

(ii) $k \neq 0$, $W = W(\tilde{\eta})$, $\tilde{U} = 0$.

Then the system (1.1) have the solutions

$$\begin{aligned}
&v = kx + \frac{\gamma c_2}{(\gamma+1)c_1}, && u = 0,\\
&p = \frac{c_2}{(\gamma+1)}, && \rho = \tfrac{(\gamma+1)c_1^2}{\gamma c_2},
\end{aligned}$$

where $c_i (i = 1\text{–}4)$ are nonzero constants.

Case 6. In order to achieve the reduction of the autonomous form, we merely define

$$E_1 = \partial_x$$
$$E_2 = \partial_y + tk\,\partial_x + k\,\partial_u,$$

whereupon X_3 and X_4 commute with X_2.

Following this result, we observe that Eqs. (2.8) and (2.12) are satisfied, but the transformation linking the original system (1.1) to the transformed system (2.8) is given by

$$\begin{aligned} &\tilde{T} = t, && p = \tilde{P}_0,\\ &\tilde{\xi} = x - kty, && u = \tilde{U} + ky,\\ &\tilde{\eta} = y, && v = \tilde{R},\\ &\rho = \tilde{H}. && \end{aligned}$$

Now we take $k \neq 0$, $W = W(\tilde{\xi})$ or $(W = W(\tilde{T}))$, $\tilde{R} = 0$ in order to obtain particular solutions and after some calculations we have

$$\begin{aligned} &u = ky + A, && \rho = Q,\\ &v = 0, && p = E. \end{aligned}$$

REFERENCES

Ames, W. F., Donato, A., and Nucci, M. C. (1989). Analysis of the threadline equations, In *Nonlinear Wave Motion*, A. Jeffrey, ed., Longman Scientific and Technical, p. 1–10. Pitman Monograph, No. 43.

Donato, A. (1992). Nonlinear waves. In *Nonlinear Equations in the Applied Science*, W. F. Ames and C. Rogers, eds., Academic Press, New York.

Donato, A. and Oliveri, F. (1993). Reduction to autonomous form by group analysis and exact solutions of axisymmetric MHD equations. *Mathematical Computational Modelling* **18**(10), 83–90.

Doyle, P. W. (1999). Symmetry and ordinary differential constraints. *International Journal of Non-Linear Mechanics* **34**, 1089–1102.

Feinsilver, P., Franz, U., and Schott, R. (2000). On solving evolution equations on Lie groups. *Journal of Physics A: Mathematical and General* **33**(12), 2419–2435.

Grigorycv, Yu. N., Meleshko, S. V., and Sattayatham, P. (1999). Classification of invariant solutions of the Boltzmann equation. *Journal of Physics A: Mathematical and General* **32**, 337–343.

Manganaro, N. and Oliveri, F. (1989). Group analysis approach in magnetohydrodynamics weak discontinuity propagation in a non-constant state. *Meccanica* **24**, 71–78.

Oliveri, F. and Paola, M. (1999). Special exact solutions to the equations of perfect gases through Lie group analysis and substitution principles. *International Journal of Non-Linear Mechanics* **34**, 1077–1087.

Olver, P. J. (1968). *Applications of Lie Groups to Differential Equations*, Springer, New York.

Ovsiannikov, L. V. (1982). *Group Analysis of Differential Equations*, Academic Press, New York.

Petrs, J. E. (1993). Applications of the group of equations of motion of polytropic gas. *International Journal of Non-Linear Mechanics* **28**(6), 663–675.

Sedove, L. I. (1959). *Similarity and Dimensional Methods in Mechanics*, Academic Press, New York.